International Journal of Theoretical Physics, Vol. 43, Nos. 7/8, August 2004 (© 2004)

States and Structure of von Neumann Algebras

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We summarize and deepen recent results on the interplay between properties of states and the structure of von Neumann algebras. We treat Jauch–Piron states and the concept of independence in noncommutative probability theory.

KEY WORDS: states on von Neumann algebras; Jauch–Piron states; independence of algebras.

1. JAUCH–PIRON STATES, σ-ADDITIVITY, AND FACIAL STRUCTURE

In this part we analyze connection between the Jauch–Piron property, σ-additivity of states and facial structure of duals of von Neumann algebras. (For basic facts on von Neumann algebras we refer to Kadison and Ringrose, 1983.) Let M be a von Neumann algebra with the projection lattice $P(M)$. A state ϱ (i.e. a positive normalized functional) on M is called Jauch–Piron if $\varrho(e \vee f) = 0$ whenever e and f are projections in M with $\varrho(e) = \varrho(f) = 0$. (The symbol $e \vee f$ stands for the supremum of projections e, f.) In the physical interpretation, projection e represents a random event of the system given by the algebra M. The linear functional ϱ describes probability distributions of all observables. The value $\varrho(e)$ is probability of event e in state ϱ of the system. The Jauch–Piron property now postulates that the events with zero probability obey the same law as in classical probability theory: if events e and f have both probability zero then the probability that event e or f occurs is again zero. Not all states on von Neumann algebras have this property (Amann, 1989; Bunce and Hamhalter, 1994). That is why the concept of Jauch–Piron state has received a great deal of attention. It was discovered and first studied by Jauch and Piron (Jauch, 1968; Jauch and Piron, 1965, 1969) in connection with propositional calculus of quantum mechanics and the theory of hidden variables. The first results on Jauch–Piron states on operator algebras were obtained by G. Rütimann (1977). A nice characterization of pure

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Jauch–Piron states on algebras on separable spaces has been given by A. Amann (1989). We have characterized the Jauch–Piron property in terms of functional analytic properties of states on von Neumann algebras and JW algebras in a series of papers (Bunce and Hamhalter, 1994, 1996, 2000; Hamhalter, 1993). This study has revealed an interesting fact that continuity properties of Jauch–Piron states are intimately connected with the lattice-theoretic properties of the projection lattice. In this way, the concept of Jauch–Piron state, introduced solely on physical grounds, has also produced a harvest of results for pure operator algebra theory. In this section, we would like to comment on the relationship between σ-additivity of states and the Jauch–Piron property. A state ϱ on a von Neumann algebra M is said to be σ-additive if $\varrho(\sum_{n=1}^{\infty} e_n) = \sum_{n=1}^{\infty} \varrho(e_n)$, whenever (e_n) is a sequence of orthogonal projections in M. If M is σ-finite (i.e. if any system of mutually orthogonal projections in M is at most countable) then ϱ is σ-additive if and only if ϱ is normal. (A state is called normal if it is continuous with respect to w^*-topology.) By using the spectral theorem, it is not difficult to show that any σ-additive state is Jauch–Piron (Bunce and Hamhalter, 1994). On the other hand, the reverse implication is far from being true. For example, let M be a von Neumann algebra acting on a separable infinite-dimensional Hilbert space H. Given an orthonormal basis (ξ_n) of H we can define a state $\varrho_1 = \sum_{n=1}^{\infty} \frac{1}{2^n}\omega_n$, where $\omega_n(x) = (x\xi_n, \xi_n)$ for all $x \in M$. It can be easily verified that the state ϱ_1 is σ-additive and faithful. (A faithful state attains nonzero value on each nonzero positive element.) As H has infinite dimension, we can find a state ϱ_2 vanishing on all one-dimensional projections. Consider now the state $\varrho = \frac{1}{2}(\varrho_1 + \varrho_2)$. The state ϱ is again faithful and thereby Jauch–Piron. Nevertheless, ϱ is not σ-additive because for one-dimensional projections e_n projecting onto $sp\{\xi_n\}$ we get, $\sum_{n=1}^{\infty} e_n = 1$, and so

$$
\begin{aligned}
1 = \varrho\left(\sum_{n=1}^{\infty} e_n\right) &= \frac{1}{2}\varrho_1\left(\sum_{n=1}^{\infty} e_n\right) + \frac{1}{2}\varrho_2\left(\sum_{n=1}^{\infty} e_n\right) \\
&= \frac{1}{2}\sum_{n=1}^{\infty} \varrho_1(e_n) + \frac{1}{2} \neq \frac{1}{2}\sum_{n=1}^{\infty} \varrho_1(e_n) = \sum_{n=1}^{\infty} \varrho(e_n).
\end{aligned}
$$

Another natural example is given by a tracial state. A tracial state ϱ is a state satisfying $\varrho(x^*x) = \varrho(xx^*)$ for all $x \in M$. Any trace is Jauch–Piron, but it need not be σ-additive in general (Bunce and Hamhalter, 1994). So examples of non-σ-additive Jauch–Piron states are generic. However, if we combine the Jauch–Piron condition with some geometric properties of states, we can enforce σ-additivity. A prototype result of this type, proved by A. Amann (1989) in separable case and in (Bunce and Hamhalter, 1994) in general case, is the following theorem.

Theorem 1.1. *Let M be a von Neumann algebra without nonzero abelian direct summand. A pure state on M is Jauch–Piron if and only if it is σ-additive.*

In order to extend this result, we introduce a few concepts. Denote by $S(M)$ the convex set of all states on a von Neumann algebra M. We shall call $S(M)$ the *state space* of M. Endowed with w^*-topology $S(M)$ is compact. A subset $F \subset S(M)$ is called *face* if F is convex and $\varrho_1, \varrho_2 \in F$ whenever $\frac{1}{2}(\varrho_1 + \varrho_2) \in F$. Singleton set $\{\varrho\}$ is a face if and only if ϱ is a pure state. Moreover, we say that a face F is a *split face* if there is a face $F^\#$ (called *complementary face*) such that each $\varrho \in S(M)$ is a unique convex combination of states $\varrho_1 \in F$ and $\varrho_2 \in F^\#$. There are deep results characterizing faces of the state space in terms of projections in the double dual of the algebra in question (see e.g. Alfsen and Schultz, 1976). All norm closed faces of $S(M)$ are of the form $\{\varrho \in S(M) | \varrho(p) = 1\}$, where p is a projection in the double dual M^{**}. All split faces are of the form $\{\varrho \in S(M) | \varrho(z) = 1\}$, where z is a central projection in M^{**}. The following examples of split faces will be important in the sequel: $S_\sigma(M)$, the set of all σ-additive states, and $S_{\mathrm{n}}(M)$, the set of all normal states. We shall study the position of the set $S_{\mathrm{J}}(M)$ of all Jauch–Piron states on M and the split face $S\sigma(M)$. We have already seen that $S_\sigma \subset S_{\mathrm{J}}(M)$. It can be easily verified that $S_{\mathrm{J}}(M)$ is a convex set and that any extreme point of $S_{\mathrm{J}}(M)$ is simultaneously an extreme point of the whole state space, i.e. a pure state (Bunce and Hamhalter, 2000). However, unlike $S_\sigma(M)$ the set $S_{\mathrm{J}}(M)$ is only exceptionally norm closed or a split face as the following theorem shows.

Theorem 1.2. (Bunce and Hamhalter (2000)) *For a von Neumann algebra M the following statements are equivalent:*

(i) *$S_J(M)$ is a face.*
(ii) *$S_J(M)$ is norm closed.*
(iii) *$S_J(M) = S(M)$ (all states are Jauch–Piron).*
(iv) *$M = M_1 \oplus M_2$, where M_1 is abelian von Neumann algebra and M_2 has finite dimension.*

It demonstrates how different is the σ-additivity and the Jauch–Piron property. If $S_{\mathrm{J}}(M)$ is a proper subset of $S(M)$ then $S_{\mathrm{J}}(M)$ is never a face and it is never norm closed. Surprisingly, making norm closure of $S_{\mathrm{J}}(M)$ we always get even split face.

Theorem 1.3. *(Bunce and Hamhalter (2000)) Let M be a von Neumann algebra. Then $\overline{S_J(M)}$ is a split face. Its complementary face*

$$\overline{S_J(M)}^\# \subset \{\varrho \in S(M) | \varrho(e) = 0 \quad \text{for all} \quad \sigma\text{-finite projections } e\}.$$

The second part of the previous theorem says that the complementary face of $\overline{S_{\mathrm{J}}(M)}$ consists of extremely discontinuous states. This is due to the fact that any projection is a sum of orthogonal σ-finite subprojections. In other words, the closure of the Jauch–Piron state space contains all reasonably continuous states.

For example, the set of all Jauch–Piron states is already dense in the whole state space provided that the algebra is σ-finite. The analysis of the facial structure of the Jauch–Piron state space has led to the following criterion of σ-additivity in terms of the Jauch–Piron property. Given a state ϱ on M and $a \in M$ with $\varrho(a^*a) \neq 0$ we define the transformed state ϱ_a by setting

$$\varrho_a(x) = \frac{\varrho(a^*xa)}{\varrho(a^*a)}, \quad x \in M.$$

In case of M being abelian, the set of all transformed states of ϱ corresponds to the set of all measures absolutely continuous with respect to ϱ.

Theorem 1.4. (Bunce and Hamhalter (2000)) *Let ϱ be a state on a von Neumann algebra M with no nonzero abelian direct summand. The following conditions are equivalent:*

(i) *ϱ is σ-additive.*
(ii) *The norm closure of the set of transformed states of ϱ consists of Jauch–Piron states.*
(iii) *The split face generated by ϱ consists of Jauch–Piron states.*

Thus, an individual Jauch–Piron state need not be σ-additive. However, if we require that all states obtained by manipulating with the physical system initially prepared in a state ϱ are Jauch–Piron, then ϱ has to be σ-additive. This may be seen as justification of σ-additivity on physical grounds. By choosing ϱ to be a pure state Theorem 1.4 reduces to Theorem 1.1. Indeed, the split face generated by a pure state ϱ is the set of all unitary transforms of ϱ. As the unitary transform preserves the Jauch–Piron property we see by equivalence of (i) and (iii) in the previous Theorem that Theorem 1.1 holds. Besides, Theorem 1.4 has the following consequence, saying that the set $S_\sigma(M)$ is uniquely determined by the Jauch–Piron state space $S_{\mathrm{J}}(M)$.

Corollary 1.5. (Bunce and Hamhalter (2000)) *If M is a von Neumann algebra with no nonzero abelian direct summand then $S_\sigma(M)$, the split face of all σ-additive states on M, is the largest split face contained in $S_J(M)$.*

Another result showing that σ-additivity is determined by the structure of Jauch–Piron states is the following result. (By the symbol π^* we mean the adjoint map of a linear map π.)

Theorem 1.6. *Let M and N be von Neumann algebras, M having no nonzero abelian direct summand. Let $\pi : M \to N$ be a $*$-homomorphism. Then π is*

σ-additive if and only if

$$\pi^*(S_J(N)) \subset S_J(M). \tag{1}$$

Proof: It has been proved in (Bunce and Hamhalter, 1995) that π is σ-additive if and only if π is a lattice homomorphism i.e.

$$\pi(e \vee f) = \pi(e) \vee \pi(f) \quad \textit{for all projections } e, f \in M. \tag{2}$$

We show that (2) is equivalent to (1). Suppose that π has the property (1). It means that $\varphi \circ \pi$ is a Jauch–Piron state on M whenever φ is a Jauch–Piron state on N. Assume, for a contradiction, that

$$\pi(e \vee f) \neq \pi(e) \vee \pi(f)$$

for some projections $e, f \in P(M)$. As $\pi(e) \vee \pi(f) \leq \pi(e \vee f)$ it means that the projection $p = \pi(e \vee f) - (\pi(e) \vee \pi(f))$ is nonzero. Therefore, we can find a normal state ψ with $\psi(p) = 1$. It implies that $\psi(\pi(e) \vee \pi(f)) = 0$ and so $\psi(\pi(e)) = \psi(\pi(f)) = 0$. The state ψ is normal and therefore Jauch–Piron. Nevertheless, for the state $\psi \circ \pi$ we have

$$(\psi \circ \pi)(e \vee f) = 1 \quad \text{while} \quad \psi \circ \pi(e) = \psi \circ \pi(f) = 0,$$

contradicting (1). The implication (2)$\Longrightarrow$(1) is obvious. The proof is completed. □

We now summarize our discussion in a form of a few equivalent conditions characterizing the σ-additivity in terms of physically plausible conditions.

Theorem 1.7. *Let ϱ be a state on a von Neumann algebra M having no nonzero abelian direct summand. We denote by π_ϱ the G.N.S. representation of ϱ and by M_ϱ the von Neumann algebra generated by $\pi_\varrho(M)$. The following conditions are equivalent.*

(i) *ϱ is σ-additive.*
(ii) *The norm closure of the set of all transformations of ϱ consists of Jauch–Piron states.*
(iii) *The G.N.S. representation π_ϱ is σ-additive.*
(iv) *$\varphi \circ \pi_\varrho$ is a Jauch–Piron state whenever φ is a Jauch–Piron state on M_ϱ.*
(v) *$\varphi \circ \pi_\varrho$ is a Jauch–Piron state whenever φ is a normal state on M_ϱ.*
(vi) *$\varphi \circ \pi_\varrho$ is a Jauch–Piron state whenever φ is a σ-additive state on M_ϱ.*
(vii) *$\varphi \circ \pi_\varrho$ is a Jauch–Piron state whenever φ is a vector state on M_ϱ.*

Proof: Equivalence of conditions (i) and (ii) has been established earlier.

Let us verify that (i) is equivalent to (iii). It is obvious that ϱ is σ-additive provided that π_ϱ is σ-additive. Conversely, suppose that ϱ is σ-additive. Let (e_n)

be an increasing sequence of projections in M such that $e_n \nearrow e$. Let $(H_\varrho, \pi_\varrho, h_\varrho)$ be G.N.S. data of ϱ, i.e. $h_\varrho \in H_\varrho$ is a unit vector such that

$$\varrho(a) = (\pi_\varrho(a)h_\varrho, h_\varrho) \quad \text{for } all\ a \in M.$$

For each unitary $u \in M$ one gets

$$(\pi_\varrho(e_n)\pi_\varrho(u)h_\varrho, \pi_\varrho(u)h_\varrho) = \varrho(u^*e_nu) \nearrow \varrho(u^*eu) = (\pi_\varrho(e)\pi_\varrho(u)h_\varrho, \pi_\varrho(u)h_\varrho)$$

as $n \to \infty$. In other words, as $\pi_\varrho(e - e_n)$ is a projection, we obtain

$$\begin{aligned} ||\pi_\varrho(e - e_n)\pi_\varrho(u)h_\varrho\|^2 &= (\pi_\varrho(e - e_n)\pi_\varrho(u)h_\varrho, \pi_\varrho(u)h_\varrho) \\ &= \varrho(u^*eu) - \varrho(u^*e_nu) \to 0 \quad as\, n \to \infty. \end{aligned} \tag{3}$$

As M is a linear span of its unitary group, the linear span of the set $\{\pi_\varrho(u)h_\varrho | u$ is unitary$\}$ is dense in H_ϱ. Hence, by (3) $\|\pi_\varrho(e - e_n)h\| \to 0$ for each $h \in H_\varrho$. It means that $(\pi_\varrho(e_n))$ converges to $\pi_\varrho(e)$ in the strong operator topology and so $\pi_\varrho(e_n) \nearrow \pi_\varrho(e)$. Thus, π_ϱ is σ-additive. The equivalence of conditions (iv)–(vii) can be derived in the same way as in the proof of Theorem 1.6. The proof is completed. □

The result on σ-additivity of Jauch–Piron pure states which was a starting point of our discussion can also be extended to much general states called factor states. A state ϱ on a von Neumann algebra M is called a *factor state* if the weak operator closure of $\pi_\varrho(M)$ is a factor, where π_ϱ is a G.N.S. representation of ϱ. The set of factor states, sometimes called the factor spectrum, plays an important role in the structure theory of von Neumann algebras and comprises e.g. pure states and special convex combinations of pure states. Factor states are characterized as being concentrated on atomic central projections in the double dual: a state ϱ on M is a factor state if and only if $\varrho(z) = 1$ for some minimal central projection z in M^{**}. It has turned out that more or less all factor states on properly infinite von Neumann algebras are σ-additive.

Theorem 1.8. (Bunce and Hamhalter (2000)) *Let M be a properly infinite von Neumann algebra. Suppose that M is σ-finite or the continuum hypothesis is true. Then all Jauch–Piron states are σ-additive.*

This result does not hold for finite algebras. For example, let $M = Z \otimes M_n$ be a tensor product of an infinite-dimensional abelian von Neumann algebra Z and the algebra M_n of all $n \times n$ matrices. Let $\varrho = \varrho_1 \otimes \tau$, where τ is the normalized trace and ϱ_1 is a singular state. (A state is called singular if it belongs to the complementary split face $S_n^\#(M)$.) Then ϱ is an example of non-σ-additive Jauch–Piron type I factor state. (The factor state is called type I if the algebra generated by its G.N.S. representation is a type I factor.) In this regard, the criterion of σ-additivity in Theorem 1.4 can be simplified as follows.

Proposition 1.9. *Let ϱ be a type I factor state on a von Neumann algebra M. Then ϱ is σ-additive if and only if the split face F_ϱ generated by ϱ contains at least one Jauch–Piron state which is an extreme point of the set $S_J(M)$.*

Proof: Suppose that $\varphi \in F_\varrho$ is an extreme point of $S_J(M)$. By (Bunce and Hamhalter, 2000) φ is a pure state, which implies that φ is σ-additive (Theorem 1.1). So the split face F_ϱ has nonzero intersection with the split face $S_\sigma(M)$. As the split faces are represented by central projections in M^{**} we can write $F_\varrho = \{\psi \in S(M) | \psi(z_\varrho) = 1\}$, $S_\sigma(M) = \{\psi \in S(M) | \psi(z_\sigma) = 1\}$, where z_ϱ and z_σ is a minimal central projection and and a central projection in M^{**}, respectively. So either $z_\varrho \leq z_\sigma$ or $z_\sigma z_\varrho = 0$. As the second possibility is excluded by the assumption we see that $\varrho \in F_\varrho \subset S_\sigma(M)$.

On the other hand, if ϱ is σ-additive then F_ϱ consists of σ-additive states and it contains a pure state because $\overline{\pi_\varrho(M)} \cong z_\varrho M^{**}$ is a type I factor.

The proof is complete. □

An analysis of the Jauch–Piron property for factor states on homogeneous type I algebras has led to the following result:

Theorem 1.10. (Bunce and Hamhalter (2000)) *Let $M = C(X) \otimes M_n$, where $2 \leq n < \infty$, $C(X)$ is algebra of continuous functions on compact hyperstonean space, and let ϱ be a factor state on M:*

(i) *If ϱ is a Jauch–Piron state and is not faithful on the center of M, then ϱ is σ-additive.*
(ii) *If ϱ is faithful on the center of M, then ϱ is a Jauch–Piron state.*

Theorems 1.8 and 1.10 demonstrate the fact that for highly noncommutative von Neumann algebras the algebraic and functional analytic properties of states are better connected then in case of algebras retaining some properties of abelian algebras.

2. INDEPENDENCE OF VON NEUMANN ALGEBRAS

In this section, we would like to consider the question of independence of von Neumann algebras. In classical probability theory, two probability spaces

$(X, \mathcal{A}_1, \mu_1)$ and $(Y, \mathcal{A}_2, \mu_2)$ living in a probability space $(Z, \mathcal{A}, \mu)$ are called *independent* if $(Z, \mathcal{A}, \mu)$ can be organized as the product space

$$(X \times Y, \mathcal{A}_1 \times \mathcal{A}_2, \mu_1 \times \mu_2).$$

Especially, for independent random variables f and g corresponding to systems X and Y, respectively, we have well-known principle for expectation value $E : E(fg) = E(f)E(g)$. The independence is a well-established concept in classical probability theory. Unlike this, there are many mutually nonequivalent generalization of this notion in noncommutative probability theory arising mostly in quantum filed theory (Baumgärtel and Wollenberg, 1992; Baumgärtel, 1995; Goldstein *et al.*, 1999; Florig and Summers, 1997; Haag, 1996; Haag and Kastler, 1964; Hamhalter, 1997, 1998, 2001, to appear; Horudzij, 1990; Kraus, 1995; Redei, 1995a,b, 1998; Roos, 1970; Summers, 1988, 1990). In our discussion, we shall focus mainly on extension type independence conditions and their interrelations. Let A and B be von Neumann subalgebras in a von Neumann algebra M. (We will always assume that subalgebra contains the unit of larger algebra.) The pair (A, B) is said to be *C^*-independent* if for any state φ_1 on A and for any state φ_2 on B there is a state φ on M extending both φ_1 and φ_2. This concept was proposed by Haag and Kastler (1964) in the context of axiomatic quantum field theory. An analogous notion in the category of W^*-algebras is the W^*-independence. The pair (A, B) is said to be *W^*-independent* if for normal states φ_1 and φ_2 on A and B, respectively, there is a normal state φ on M extending both φ_1 and φ_2. In classical measure theory, the W^*-independence amounts to a question of the existence of a common completely additive extension of two completely additive measures prescribed on subsystems. The problem of the relationship between C^*- and W^*-independence has been settled only recently (Florig and Summers, 1997; Hamhalter, 1997, in press). First of all, W^*-independence always implies C^*-independence. Indeed, as the set of normal states is w^*-dense in the state space, W^*-independence implies the existence of common extensions for pairs of states forming w^*-dense set in the product of state spaces of local algebras. As the state space is compact in the w^*-topology we see that it is enough to imply C^*-independence. The set of normal states has different character then the state space (it is not compact, etc.). For this reason W^*-independence seems to be, at first glance, different from C^*-independence. Nevertheless, surprising result of Florig and Summers (1997) has shown that C^*-independence coincides with W^*-independence for mutually commuting C^* algebras acting on separable Hilbert spaces. Representation on separable space is a basic ingredient of their proof. The question of validity of this theorem for general algebras remains open. In subsequent analysis of independence of subalgebras in general position, we have proved that W^*- and C^*-independence split up for all essentially noncommutative algebras.

Theorem 2.1. *(Hamhalter (in press)) Let M be a von Neumann algebra. The following conditions are equivalent:*

(i) *M has an infinite-dimensional nonabelian direct summand.*
(ii) *There are finite-dimensional von Neumann subalgebras A and B in M which are C^*-independent but not W^*-independent.*

Even, it can be proved that for any von Neumann algebra with infinite-dimensional nonabelian direct summand, we can find two projections $p, q \in M$ (of course not commuting) such that two-dimensional subalgebras

$$A = sp\{p, 1 - p\}, \quad B = \{q, 1 - q\}$$

are C^*-independent but not W^*-independent. Note that condition (i) in Theorem 2.1 is equivalent to the existence of state on M which is not Jauch–Piron. This fact is not a coincidence, the construction of a state which is not Jauch–Piron is an important step in constructing algebras A and B earlier. However, despite of this negative result, the position of C^*- and W^*-independence is more delicate. The following result shows that C^*-independence is very close to W^*-independence.

Theorem 2.2. *Let A and B be C^*-independent von Neumann subalgebras in a von Neumann algebra M. Then for any $\varepsilon > 0$ and for all normal states φ_1 and φ_2 on A and B, respectively, there is a normal state φ on M such that*

$$\|\varphi|A - \varphi_1\| < \varepsilon \quad \text{and} \quad \|\varphi|B - \varphi_2\| < \varepsilon.$$

Therefore, C^*-independence is strong enough to ensure that the pairs of normal states on local algebras A and B that do admit common normal extension over M are norm dense in the product of normal state spaces. Because of ubiquitous error in measurement, Theorem 2.2 says that we cannot distinguish between C^*- and W^*-independence in a real situation. On the other hand, W^*- and C^*-independence considerably differ in the following respect. It was proved by H. Roos (1970) that for mutually commuting C^*-independent C^* algebras A and B in a C^* algebra C one can find for any pair of states φ_1 on A and φ_2 on B a state φ on C (called *product state*) such that

$$\varphi(ab) = \varphi_1(a)\varphi_2(b) \quad \text{for all} \quad a \in A, b \in B.$$

The product state embodies the well-known law for expectation value of product of independent variables. Although for mutually commuting von Neumann algebras on separable space C^*-independence coincide with W^*-independence, we can seldom find simultaneous product *normal* extension. For example, it is known that the von Neumann algebra M generated by mutually commuting von Neumann algebras A and B is isomorphic to the tensor product $A\overline{\otimes}B$ whenever there is at least one normal product state on M with central support 1 (D'Antoni and Longo, 1983; Takesaki, 1958). Let us now take a type III factor A on a separable Hilbert space H. Then the pair (A, A'), where A' denotes the commutant of A in the algebra $B(H)$ of all bounded operators on H, is both C^*- and W^*-independent. The von Neumann algebra generated by A and A' is a type I factor $B(H)$. As the tensor product of type III factors is again type III factor, we have that $A\overline{\otimes}A'$ is not isomorphic to the algebra generated by A and A'. In this case, all pairs of normal states on A and A' extend simultaneously to normal states on $B(H)$, but none of these extensions is a product state. On the other hand, any pair of normal states has common product extension. Let us remark that this situation is typical in quantum field theory. This advocates the use of nonnormal states in the C^* algebraic approach approach to quantum filed theory (Haag, 1996; Haag and Kastler, 1964).

ACKNOWLEDGEMENT

The author would like to express his gratitude to the Alexander von Humboldt Foundation for the support of his research the result of which are contained in this paper. He would also like to thank to the Grant Agency of the Czech Republic, Grant No. 201/00/0331, and Czech Technical University, Grant No. MSM 210000010, for supporting his research activity.

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